Quantum target detection using entangled photons

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We investigate performances of pure continuous variable states in discriminating thermal and identity channels by comparing their M-copy error probability bounds. This offers us a simplified mathematical analysis for quantum target detection with slightly modified features: the object – if it is present – perfectly reflects the signal beam irradiating it, while thermal noise photons are returned to the receiver in its absence. This model facilitates us to obtain analytic results on error-probability bounds i.e., the quantum Chernoff bound and the lower bound constructed from the Bhattacharya bound on M-copy discrimination error-probabilities of some important quantum states, like photon number states, N00N states, coherent states and the entangled photons obtained from spontaneous parametric down conversion (SPDC). Comparing the M-copy error-bounds, we identify that N00N states indeed offer enhanced sensitivity than the photon number state system, when average signal photon number is small compared to the thermal noise level. However, in the high signal-to-noise scenario, N00N states fail to be advantageous than the photon number states. Entangled SPDC photon pairs too outperform conventional coherent state system in the low signal-to-noise case. On the other hand, conventional coherent state system surpasses the performance sensitivity offered by entangled photon pair, when the signal intensity is much above that of thermal noise. We find an analogous performance regime in the lossy target detection (where the target is modeled as a weakly reflecting object) in a high signal-to-noise scenario.

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I. INTRODUCTION

Entangled states generally offer enhanced sensitivity over unentangled ones in channel discrimination. More specifically, it is shown that minimum error-probability in distinguishing two generalized Pauli channels in any dimension is acheived by employing maximally entangled states as input states [1]. Extending these ideas, Lloyd [2] proposed his quantum illumination scheme for target detection: Single photons (signal) from a maximally entangled pair are transmitted towards the target (which is modeled as a weak reflector with reflectivity $\kappa << 1$) immersed in thermal noise. The received light is then measured jointly with the retained idler photon. When the object is absent, only thermal radiation is returned and the presence of the object corresponds to a lossy return of the signal radiation combined with the thermal noise. The efficiency of target detection i.e., the sensitivity of discriminating the returned light in the two situations, when the target is absent (channel 0) or present (channel 1) [3] is established – with the help of quantum Chernoff bound [4] on error exponents – to be substantially enhanced with an entangled photon transmitter, when compared with the performance of an unentangled single photon transmitter. However, the analysis in Ref. [2] was confined to the single photon regime and more recently [5], a full Gaussian state analysis confirmed that for noisy Gaussian channels, a low brightness *quantum illumination* – using entangled photons obtained from a continuous wave SPDC – is indeed advantageous compared to that with a coherent light. It is further realized that [6] the quantum illumination system of Ref. [2] – which was restricted to the vacuum plus one photon manifold – *does not* improve the performance over that of a conventional coherent state transmitter in the low noise regime. The dramatic entanglement induced 6 dB error exponent gain over the classical coherent state transmitter system [5] however persists in the low brightness, lossy, noisy regime. A receiver design achieving up to 3 dB gain in error exponent has also been proposed [7] for the quantum illumination system with a low intensity transmitter operating in a highly lossy, noisy regime.

These recent investigations on quantum illumination system to detect a low reflectivity target form the motivation to explore a simpler mathematical model that captures and elucidates the role of continuous variable entanglement in discrimination. To this end, we begin by noting that a $d \times d$ pure maximally entangled state exhibits an unambiguous improvement in discriminating the identity and the completely depolarizing channels over an unentangled d dimensional state [1]. It would be natural to seek a similar mathematical model for target detection, where the object (when present) acts as a perfect mirror with reflectivity $\kappa = 1$ and thus, corresponds to identity channel for any input state of radiation, whereas a thermal channel represents its absence. In this paper, we analyze contrasting regimes of performance for target detection in this scenario using entangled photons, compared to unentangled ones. With this

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background, a re-look at full Gaussian analysis [5] of the lossy, noisy situation, employing coherent and entangled SPDC photon systems reveals analogous behavior and it is found that coherent light outperforms entangled photon system when signal intensity exceeds far above that of thermal noise.

The paper is organized in four sections. In Sec. II, preliminary ideas on channel discrimination are given. An example demonstrating the performance advantage of $d\times d$ maximally entangled pure state and the Werner state over that of an arbitrary d-dimensional single party pure state in discriminating identity and completely depolarizing channels is discussed. This is followed by Sec. III where discrimination of thermal and identity channels with pure states of photons is reported. This serves as a simple model for quantum target detection, where signal light irradiating the object (when it is present) is reflected perfectly i.e., without any loss, while a thermal radiation is returned in its absence. This model is useful as it allows explicit analytic results on error-probabilities or upper (quantum Chernoff bound) and lower bounds on error-probabilities when M repeated uses of the transmitted photon states is considered. We compare the performances of (A) photon number states vs. N00N states, and (B) coherent light vs. two-mode entangled photons obtained from SPDC process. The contrasting performance behavior identified in this model prompts us to include a brief discussion on target detection in a lossy, noisy scenario with high signal-to-noise ratio. In Sec. IV, we give a summary of our results.

II. PRELIMINARY IDEAS

Let us consider the problem of quantum state discrimination, where one has to distinguish between two possible states ρ_0, ρ_1 of a quantum system. When both the quantum states are equally probable and M copies of the states available for measurement, the probability of error is given by [8],

$$P_e^{(M)}=\frac{1}{2}\left(1-\frac{1}{2}||\rho_0^{\otimes M}-\rho_1^{\otimes M}||_1\right) \tag{1}$$

where $||A||_1=\mathrm{Tr}[\sqrt{A^\dagger A}]$.

The question of distinguishing two channels Φ_0 and Φ_1 with a given input state ρ can be reformulated in terms of discrimination of the quantum states ρ_0 and ρ_1, when they turn out to be the output states of the channels 0, 1 respectively. The single copy error-probability for channel discrimination has the form,

$$\begin{aligned} P_e^{(1)} &= \frac{1}{2}\left(1-\frac{1}{2}||\Phi_0(\rho)-\Phi_1(\rho)||_1\right) \\ &= \frac{1}{2}\left(1-\frac{1}{2}||\rho_0-\rho_1||_1\right). \end{aligned} \tag{2}$$

When the input state is a composite bipartite quantum system, with the channel affecting only one part of the state, the single-shot error-probability is expressed as,

$$P_e^{(1)}=\frac{1}{2}\left(1-\frac{1}{2}||(\Phi_0\otimes I)\rho-(\Phi_1\otimes I)\rho||_1\right). \tag{3}$$

In the simple example, where a completely depolarizing channel and an identity channel – labeled respectively as channel 0 and channel 1 – are to be discriminated using a pure d dimensional input state $|\psi\rangle\in\mathcal{H}_d$, the output states are given by,

$$\begin{aligned} \rho_0 &= \Phi_0(\rho)=\frac{I}{d} \\ \rho_1 &= \Phi_1(\rho)=|\psi\rangle\langle\psi|. \end{aligned} \tag{4}$$

The probability of error in distinguishing ρ_0 and ρ_1 is readily found to be,

$$\begin{aligned} P^{(1)}_{e,|\psi\rangle} &= \frac{1}{2}\left(1-\frac{1}{2}\left|\left|\frac{I}{d}-|\psi\rangle\langle\psi|\right|\right|_1\right) \\ &= \frac{1}{2}\left(1-\frac{1}{2}\left[\left|\frac{1}{d}-1\right|+\frac{d-1}{d}\right]\right)=\frac{1}{2d}. \end{aligned} \tag{5}$$

With a maximally entangled $d\times d$ input state,

$$|\Psi_{AB}\rangle=\frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k_A,k_B\rangle, \tag{6}$$

we obtain,

$$\begin{aligned} \rho_0 &= (\Phi_0\otimes I)|\Psi_{AB}\rangle\langle\Psi_{AB}| \\ &= \frac{I}{d}\otimes\mathrm{Tr}_A[|\Psi_{AB}\rangle\langle\Psi_{AB}|]=\frac{I\otimes I}{d^2} \\ \text{and}\quad \rho_1 &= (\Phi_1\otimes I)|\Psi_{AB}\rangle\langle\Psi_{AB}|=|\Psi_{AB}\rangle\langle\Psi_{AB}|. \end{aligned} \tag{7}$$

The error-probability in discriminating the two channels, with a maximally entangled state is given by,

$$P^{(1)}_{e,|\Psi_{AB}\rangle}=\frac{1}{2d^2}. \tag{8}$$

So, maximally entangled states (6) reveal an enhanced performance in the discrimination of completely depolarizing and identity channels [1].

To emphasize this further, let us consider a bipartite Werner state,

$$\rho_{\rm W}=\frac{(1-x)}{d^2}\,I\otimes I+x\,|\Psi_{AB}\rangle\langle\Psi_{AB}|;\quad 0\le x\le 1, \tag{9}$$

which is entangled for $1/(d+1)<x\le 1$ [9]. We obtain the probability of error in discriminating the channels as,

$$P^{(1)}_{e,\rho_W}=\frac{d^2-x(d^2-1)}{2\,d^2}. \tag{10}$$

The bipartite Werner state clearly shows an advantage over the d- dimensional single party pure state if $x>\frac{d}{d+1}$. In other words, the performance enhancement offered by entangled states over unentangled ones is brought out explicitly in this illustrative case of channel discrimination.

In the next section this analysis is extended to investigate a simple mathematical model for quantum target detection, where we explore the sensitivity of entangled photon states vs unentangled ones in the detection of a perfectly reflecting target – which in turn reduces to discriminating thermal and identity channels.

III. DISCRIMINATION OF THERMAL AND IDENTITY CHANNELS WITH PHOTONS:

Let us imagine a quantum target detection experiment, where an optical transmitter sends light towards a region where a perfectly reflecting object is suspected to be present. The object, when present, reflects light falling on it to the receiver end. When the object is absent, the signal light passes through the region undeflected and a thermal noise radiation is returned to the receiver. Subsequently, the returned light is processed by the receiver to decide between the two hypotheses, H_0 : *object not there* and H_1 : *object there.* In other words, the receiver has to distinguish between two quantum states of light – one, the output of a thermal channel (object not there) and the other, that of an identity channel (object there). The states at the receiver are,

$$\begin{aligned}
&\text{Hypothesis 0 (object not there) :}\\
&\rho_0 = \rho_{\rm th}(N_B) = \sum_{k=0}^{\infty} \frac{N_B^k}{(N_B+1)^{k+1}} |k\rangle\langle k|,\\
&\qquad = (1-e^{-\beta}) \textstyle\sum_{k=0}^{\infty} e^{-k\beta} |k\rangle\langle k|,\\
&\qquad\quad \text{where } N_B = \frac{e^{-\beta}}{(1-e^{-\beta})},\\
&\text{Hypothesis 1 (object there) :}\\
&\rho_1 = \rho_{\rm in}. \qquad (11)
\end{aligned}$$

where $\rho_{\rm in}$ denotes the input state.

With M copies of the states available, the probability of making an incorrect decision takes its minimum value (see Eq. (1)) when a joint optimal measurement involving projectors on the positive and negative eigenspaces of the operator $\rho_0^{\otimes M} - \rho_1^{\otimes M}$ could be performed. If this measurement results in negative eigenvalues, the decision is in favour of ρ_1 (object present); otherwise, it is concluded that ρ_0 is the received state (object not there). Keeping aside the question on experimental feasibility of such optimal joint-detection leading to maximum sensitivity of making a correct decision between the two hypotheses, it is in fact a hard computational task to evaluate the trace-norm $||\rho_0^{\otimes M} - \rho_1^{\otimes M}||_1$ in order to estimate the probability of error. The method often followed in decision theory is to establish bounds on the error probability $P_e^{(M)}$ in order to get an insight on how the probability of making an incorrect decision declines with number of copies M. The error-probability is upper bounded by the quantum Chernoff bound [4]

$$P_e^{(M)} \leq P_{e,QCB}^{(M)} \equiv \frac{1}{2}\left(\min_{0\leq s\leq 1}\, {\rm Tr}[\rho_0^s \rho_1^{1-s}]\right)^M, \qquad (12)$$

which gives the asymptotic exponential error decline $\lim_{M\rightarrow\infty} P_e^{(M)} \sim \frac{1}{2}\, e^{-M\,\xi_{QCB}}$ with $\xi_{QCB} = -\min_{0\leq s\leq 1} \ln {\rm Tr}[\rho_0^s\rho_1^{1-s}]$ representing the logarithmic quantum Chernoff bound. Further, a computable lower limit on probability of error is established as

$$P_{e,LB}^{(M)} = \frac{1}{2}\left(1 - \sqrt{1 - ({\rm Tr}[\rho_0^{\frac{1}{2}}\rho_1^{\frac{1}{2}}])^{2M}}\right) \leq P_e^{(M)}, \qquad (13)$$

which is related to the Bhattacharya bound – a weaker upper bound, obtained by substituting $s = 1/2$ in (12).

In the special case, when both the states to be discriminated are pure i.e., $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and $\rho_1 = |\psi_1\rangle\langle\psi_1|$, one obtains an exact result for error-probability [10]:

$$P_e^{(M)} = \frac{1}{2}\left(1 - \sqrt{1 - |\langle\psi_0|\psi_1\rangle|^{2M}}\right). \qquad (14)$$

With only one of the states, say ρ_1, is pure, the quantum Chernoff bound is related to the fidelity:

$$P_e^{(M)} \leq P_{e,QCB}^{(M)} = \frac{1}{2}\,\langle\psi_1|\rho_0|\psi_1\rangle^M. \qquad (15)$$

(equality sign holds when the states commute with each other).

Coming back to quantum target detection with a perfectly reflecting object, it would be useful to restrict here to optical transmitters sending pure states of photons, as this scenario is more amenable to obtaining analytic results and lead to a better insight into exploring performances of some important quantum states of photons in target detection.

A. Photon number states vs N00N states:

Employing an optical transmitter, which sends photon number states $|n\rangle$ to shine the object, we obtain (see Eq. (11)),

$$\rho_0 = \rho_{\rm th}(N_B), \text{ and } \rho_1 = |n\rangle\langle n|. \qquad (16)$$

Substituting (16) in (15) and simplifying, we obtain the exact result [11] for error-probability:

$$\begin{aligned}
P_{e,n}^{(M)} &= \frac{1}{2}\left[\frac{N_B^n}{(1+N_B)^{n+1}}\right]^M\\
&= \frac{1}{2}(1-e^{-\beta})^M\, e^{-Mn\beta}. \qquad (17)
\end{aligned}$$

On the other hand, entangled pair of photons sharing a N00N state

$$|\Psi_{N00N}^{SI}\rangle = \frac{1}{\sqrt{2}}\,[|2n,0\rangle + |0,2n\rangle], \qquad (18)$$

with average photon number $\langle a_S^\dagger a_S \rangle = \langle a_I^\dagger a_I \rangle = n$ per both signal (S) and idler (I) modes, results in the following states to be distinguished by the receiver:

$$\begin{aligned}\rho_0 &= \rho_{\rm th}(N_B) \otimes {\rm Tr}_S[|\Psi^{SI}_{N00N}\rangle\langle\Psi^{SI}_{N00N}|] \\ &= \rho_{\rm th}(N_B) \otimes \frac{1}{2}[|0\rangle\langle 0| + |2n\rangle\langle 2n|] \\ \rho_1 &= |\Psi^{SI}_{N00N}\rangle\langle\Psi^{SI}_{N00N}|. \end{aligned} \tag{19}$$

We evaluate the quantum Chernoff bound on the M-shot error-probability as follows:

$$\begin{aligned} P^{(M)}_{e,QCB,N00N} &= \frac{1}{2}\left[\langle\Psi^{SI}_{N00N}|\left\{\rho_{\rm th}(N_B)\otimes\frac{1}{2}[|0\rangle\langle 0| + |2n\rangle\langle 2n|]\right\}|\Psi^{SI}_{N00N}\rangle\right]^M \\ &= \frac{1}{2}\left(\frac{N_B^n}{(1+N_B)^{n+1}}\left[\frac{1}{4}\left\{\left(\frac{1+N_B}{N_B}\right)^n + \left(\frac{1+N_B}{N_B}\right)^{-n}\right\}\right]\right)^M \\ &= \frac{1}{2}(1-e^{-\beta})^M\, e^{-Mn\beta}\, \left(\frac{\cosh(n\beta)}{2}\right)^M \end{aligned} \tag{20}$$

A comparison of (17) and (20) indicates that entangled N00N states do offer enhanced sensitivity over photon number states of same signal intensity n, when $\cosh(n\beta) < 2$ as $P^{(M)}_{e,n} > P^{(M)}_{e,QCB,N00N}$ in this case. However, the situation appears to get reversed if the signal intensity n is much larger (for a given thermal noise β) such that $\cosh(n\beta) > 2$, in which case the upper bound $P^{(M)}_{e,QCB,N00N}$ on N00N state's M-copy error probability is greater than photon number state's error probability $P^{(M)}_{e,n}$. Note that the underperformance of N00N state system holds as an exact result in the limit $M \to \infty$. One has to verify if the lower bound on error-probability (see (13)) with N00N state system too confirms this observation. In order to identify this, we first evaluate ${\rm Tr}[\rho_0^{\frac{1}{2}}\rho_1^{\frac{1}{2}}]$ for the output states (19) to be discriminated i.e.,

$$\begin{aligned} {\rm Tr}[\rho_0^{\frac{1}{2}}\rho_1^{\frac{1}{2}}] &= \langle\Psi^{SI}_{N00N}|\{\rho^{\frac{1}{2}}_{\rm th}(N_B)\otimes\frac{1}{\sqrt{2}}[|0\rangle\langle 0| + |2n\rangle\langle 2n|]\}|\Psi^{SI}_{N00N}\rangle \\ &= \frac{1}{2\sqrt{1+N_B}}\left[1 + \left(\frac{N_B}{N_B+1}\right)^n\right] \\ &= \sqrt{\frac{e^{-n\beta}(1-e^{-\beta})}{2}}\cosh(n\beta/2). \end{aligned} \tag{21}$$

to obtain the lower bound on M-copy error-probability with N00N states as,

$$P^{(M)}_{e,LB,{\rm N00N}} = \frac{1}{2}\left[1 - \sqrt{1 - \left(\sqrt{\frac{e^{-n\beta}(1-e^{-\beta})}{2}}\,\cosh(n\beta/2)\right)^{2M}}\,\right] \le P^{(M)}_{e,{\rm N00N}} \tag{22}$$

In Fig. 1, we compare the photon number state's error-probability $P^{(M)}_{e,n}$ given by (17) with upper (quantum Chernoff bound) and lower bounds on $P^{(M)}_{e,{\rm N00N}}$ of N00N

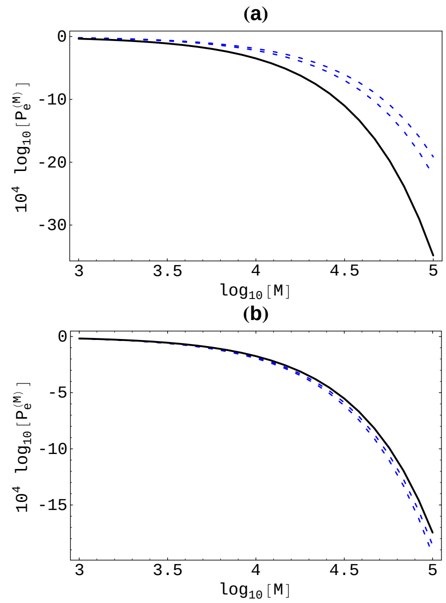

FIG. 1: (color online) Upper, lower bounds (dashed curves) on M-copy error-probability with N00N states and photon number state's error-probability (solid curve) for a thermal noise $\beta = 0.05$; photon numbers in (a) n=100 and in (b) n=20. The lower bound lies above the number state error-probability in (a) implying that N00N states are *not* advantageous over photon number states. But, with smaller number of photons (as illustrated in (b)), entangled N00N states indeed offer an enhanced sensitivity over number state system.

state (see (20), (22)) in two different cases (a) n=100 (b) n=20, for a fixed thermal noise $\beta = 0.05$ (which corresponds to average number of thermal photons $N_B \sim 20$). We find that the error-probability bounds corresponding to N00N state are higher in magnitude than the photon number state error-probability for large values of n and this provides a clear evidence that N00N states do not offer any performance enhancement over unentangled photon number states. On the other hand, N00N states offer enhanced sensitivity compared to number states, when low photon numbers n (such that $\cosh(n\beta) < 2$) are considered (here, the lower bound $P^{(M)}_{e,LB,\mathrm{N00N}}$ on N00N state's error-probability is smaller in magnitude, when compared with the error-probability $P^{(M)}_{e,n}$ of the phton number state – as illustrated in Fig. 1(b)– bringing out the advantage of N00N states over photon number states in this regime).

B. Coherent light vs two mode entangled photons from SPDC process:

Let us consider an optical transmitter sending coherent photons in the quantum state,

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{l=0}^{\infty}\frac{\alpha^l}{\sqrt{l!}}|l\rangle. \tag{23}$$

The quantum Chernoff bound on the error probabilities is simplified as follows:

$$\begin{aligned} P^{(M)}_{e,QCB,\mathrm{coh}} &= \frac{1}{2}\langle\alpha|\rho_{\mathrm{th}}(N_B)|\alpha\rangle^M \\ &= \frac{1}{2}\left(\sum_k |\langle\alpha|k\rangle|^2\ \frac{N_B^k}{(N_B+1)^{k+1}}\right)^M \\ &= \frac{1}{2}\,\frac{e^{\frac{-M\,N_S}{N_B+1}}}{(N_B+1)^M}; \qquad |\alpha|^2 = N_S. \end{aligned} \tag{24}$$

The lower bound (13) with coherent light too can be readily evaluated following similar procedure as above and we obtain,

$$\begin{aligned} P^{(M)}_{e,LB,\mathrm{coh}} &= \frac{1}{2}\left(1-\sqrt{1-\langle\alpha|\rho_{\mathrm{th}}^{\frac{1}{2}}(N_B)|\alpha\rangle^{2M}}\right) \\ &= \frac{1}{2}\left(1-\sqrt{1-\frac{e^{-2\,M\,N_S\left(1-\sqrt{\frac{N_B}{N_B+1}}\right)}}{(N_B+1)^M}}\right) \end{aligned} \tag{25}$$

Employing entangled pair of photons from SPDC, characterized by the quantum state,

$$|\Psi^{SI}_{\mathrm{SPDC}}\rangle = \sum_{k=0}^{\infty}\sqrt{\frac{N_S^k}{(N_S+1)^{k+1}}}|k_S, k_I\rangle. \tag{26}$$

where N_S denotes the average number of photons per each mode, the quantum Chernoff bound on M-shot error-probability is evaluated below:

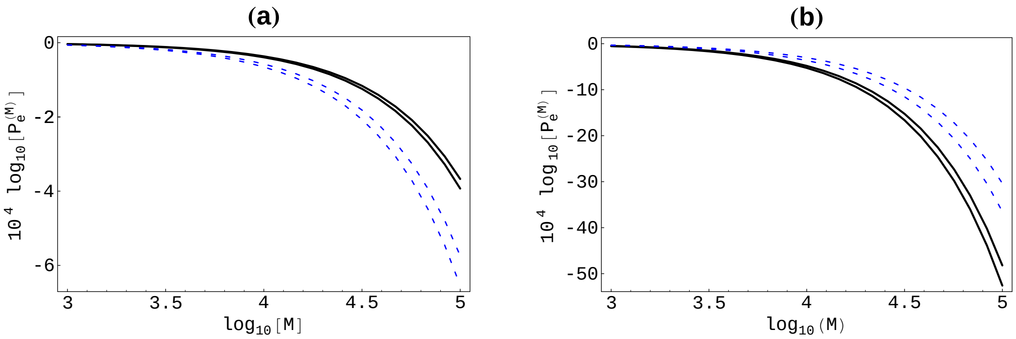

FIG. 2: (color online) Logarithms of upper and lower bounds (dashed curves) on M-shot error-probability with entangled photon pairs from SPDC source and that of coherent state system (solid curves) for (a) thermal noise $N_B = 0.75$ and $N_S = 0.5$ and in (b) $N_B = 2,\ N_S = 30$, plotted as a function of $\log_{10}[M]$. The target detection with $N_S < N_B$ in (a) is illustrative of the regime where entangled photon pairs show enhanced performance sensitivity over coherent light. But, it is seen from (b) that when $N_S >> N_B$ coherent state system is more advantageous than entangled SPDC photon pairs.

$$\begin{aligned}
P^{(M)}_{e,QCB,\mathrm{SPDC}} &= \frac{1}{2}\langle\Psi^{SI}_{\mathrm{SPDC}}|\left(\rho_{\mathrm{th}}(N_B)\otimes \mathrm{Tr}_S[|\Psi^{SI}_{\mathrm{SPDC}}\rangle\langle\Psi^{SI}_{\mathrm{SPDC}}|]\right)|\Psi^{SI}_{\mathrm{SPDC}}\rangle^M \\
&= \frac{1}{2}\langle\Psi^{SI}_{\mathrm{SPDC}}|\rho_{\mathrm{th}}(N_B)\otimes\rho_{\mathrm{th}}(N_S)|\Psi^{SI}_{\mathrm{SPDC}}\rangle^M \\
&= \frac{1}{2}\left[\frac{1}{(N_S+1)^2(N_B+1)}\sum_k \frac{N_S^{2k}N_B^k}{(N_S+1)^{2k}(N_B+1)^k}\right]^M \\
&= \frac{1}{2}\left[\frac{1}{(N_S+1)^2(N_B+1)-N_S^2N_B}\right]^M
\end{aligned} \tag{27}$$

We similarly obtain lower bound on $P^{(M)}_{e,SPDC}$ as,

$$\begin{aligned}
P^{(M)}_{e,LB,\mathrm{SPDC}} &= \frac{1}{2}\left(1-\sqrt{1-\langle\Psi^{SI}_{\mathrm{SPDC}}|\rho^{\frac{1}{2}}_{\mathrm{th}}(N_B)\otimes\rho^{\frac{1}{2}}_{\mathrm{th}}(N_S)|\Psi^{SI}_{\mathrm{SPDC}}\rangle^{2M}}\right) \\
&= \frac{1}{2}\left(1-\sqrt{1-\left[\frac{1}{\sqrt{(N_S+1)^3(N_B+1)}-\sqrt{N_S^3N_B}}\right]^{2M}}\right)
\end{aligned} \tag{28}$$

In Fig. 2 we compare the target-detection error-probability bounds of coherent state system (given by (24),(25)) with that of SPDC photon pair system (as in (27),(28)). We identify two different regimes of performance: (a) $N_S < N_B$; The error-probability upper bound is smaller in magnitude than the coherent state system's lower bound confirming enhanced performance of entangled SPDC photon pair over coherent light [5]. On the other hand, coherent state system outperforms entangled photon pair system when (b) $N_S >> N_B$, as the lower bound on target-detection error probability with entangled photons is larger in magnitude compared to the upper bound on error of the coherent state system.

At this point, it would be pertinent to take a closer look at two extreme limits, one with very bright thermal noise, $N_B \to \infty$ and the other, the weak noise limit $N_B \to 0$. In the first case, the thermal channel acts as a completely de-polarizing channel, sending equi-probable ran-

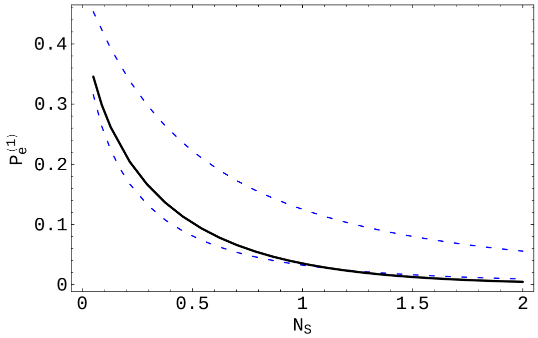

FIG. 3: (color online) A comparison of single-copy error-probability achievable with coherent state system (solid curve) with corresponding upper and lower error-probability bounds (dashed curves) with entangled photon pair system, in the weak thermal noise limit $N_B \to 0$, plotted as a function of average signal photon number N_S. Coherent state system is more advantageous compared to entangled photon pair system when the average signal photon number $N_S > 1$, as the lower bound on entangled photon single-copy-error-probability exceeds the error-probability $P^{(1)}_{e,\mathrm{coh}}$ of coherent state in the weak noise limit.

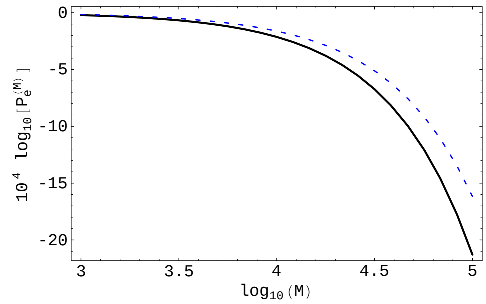

FIG. 4: (color online) A comparison of M-shot quantum Chernoff bound on error-probability achievable with coherent state system (solid curve) with the corresponding lower bound (dashed curve) associated with entangled photon pair system in the lossy (relectivity $\kappa = 0.01$), noisy (average thermal noise photons $N_B = 20$) target detection scenario using highly intense signals (with signal to noise ratio $N_S/N_B = 2000$). It may be seen that the lower bound on entangled photon error-probability lies above the upper bound on coherent state error-probability revealing that coherent state system turns out to be more advantageous compared to entangled photon pair system, with very high signal to noise ratio.

dom mixtures $\lim_{N_B\to\infty} \rho_{\rm th}(N_B) \to \dfrac{1}{N_B}\displaystyle\sum_k |k\rangle\langle k|$ as output states. The M-shot error probability of target-detection with coherent states approaches the value $P^{(M)}_{e,\mathrm{coh}} \to \frac{1}{2\,N_B^M}$. On the other hand, the quantum Chernoff bound with entangled SPDC photons tends towards $P^{(M)}_{e,QCB,\mathrm{SPDC}} \to \frac{1}{2\,N_B^M}\,\frac{1}{(2N_S+1)^{2M}}$, which is clearly smaller than the coherent state error-probability $\frac{1}{2\,N_B^M}$ in the bright noise limit. This establishes unequivocally the performance enhancement of entangled photon pairs over coherent light in the bright noise limit.

In the weak noise limit, the output of the thermal channel is a pure state $\lim_{N_B\to 0} \rho_{\rm th}(N_B) \to |0\rangle\langle 0|$. The error-probability with coherent state system in this limit is obtained (using Eq. (14)) as, $P^{(M)}_{e,\mathrm{coh}} \to \frac{1}{2}[1-\sqrt{1-e^{-2MN_s}}]$. The quantum Chernoff bound with SPDC photon pair system approaches the value $P^{(M)}_{e,QCB,\mathrm{SPDC}} \to \frac{1}{2}\,\frac{1}{(N_S+1)^{2M}}$, whereas error-probability lower bound goes as $P^{(M)}_{e,LB,\mathrm{SPDC}} \to \frac{1}{2}\,[1-\sqrt{1-1/(N_S+1)^{3M}}]$ in the low noise limit. Fig. 3 compares the low-noise-limit single-copy error bounds of coherent state and entangled photon pairs. We find that the entangled photon pair system is unlikely to offer enhanced performance over conventional coherent state system in the weak noise limit, when the average signal photon number $N_S > 1$ (as depicted in Fig. 3, the error-probability lower bound corresponding to entangled photons increases in magnitude beyond the coherent state error-probability for $N_S > 1$).

Having analyzed performance regimes where entangled SPDC photon pair system is likely or unlikely to be advantageous in quantum target detection compared to conventional coherent state system in this simpler mathematical model, it is worth revisiting the lossy, noisy scenario (where the object is modeled as a weak reflector) with a high signal-to-noise ratio. The evaluation of error-probability bounds is much involved and requires a full Gaussian analysis [5, 12] in this situation. Without going into the detailed evaluation of the error-probability bounds, we illustrate here the highlighting features in Fig. 4. We find that the coherent state's error-probability upper bound (i.e., quantum Chernoff bound – which turns out to be the Bhattacharya bound [5]) is lower than the entangled SPDC photon pair's lower bound (obtained by evaluating the bound given in (13) for the joint idler-return-mode mixed Gaussian state under both hypotheses H_0 and H_1) only when the signal intensity exceeds far above the thermal noise level ($N_S/N_B \approx 2000$ in Fig. 4). So, in the lossy ($\kappa << 1$), noisy ($N_B >> 1$) target detection scenario, the coherent state system can surpass the performance sensitivity achievable by entangled SPDC photon pair system, when a bright signal (with a large signal-to-noise ratio) is employed – this being a feature revealed by the simple mathematical model discussed above.

IV. SUMMARY

In the light of recent investigations [2, 5, 6, 7] on the advantage offered by maximally entangled SPDC photons over conventional coherent light in target detection in the lossy, noisy scenario employing low brightness signal, we have explored a simpler mathematical model elucidating the performances of pure continuous variable states in distinguishing thermal and identity channels by

evaluating the discrimination-error-probability bounds. This offers as a simple mathematical model for quantum target detection, where the object (when present) acts as a perfect mirror with reflectivity $\kappa = 1$, corresponding to identity channel for any input state or light, whereas a thermal channel signifies the absence of the object. This model facilitates analytic results on exact M-copy error-probabilities or upper (quantum Chernoff bound) and lower bounds on error-probabilities, which are explicitly evaluated here for photon number states, N00N states, coherent states and the entangled photons obtained from spontaneous parametric down conversion (SPDC). It is shown that N00N states are not advantageous over photon number states when mean number of signal photons is larger than thermal noise photons. But in the low brightness regime, N00N states indeed offer enhanced sensitivity compared to the photon number state system. Entangled SPDC photon pair is also shown to outperform conventional coherent photons in the low signal-to-noise scenario – while a contrasting behavior (i.e., coherent state system beating the performance sensitivity offered by entangled photon pair) is identified when the signal intensity exceeds far above that of the thermal noise. We have identified a similar performance regime in the lossy, noisy target detection [5], where conventional coherent radar system achieves improved sensitivity over that of the entangled photon pair system, in high signal-to-noise scenario.

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